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# THE NUCLEOLUS AND INHERITANCE OF PROPERTIES IN COMMUNICATION SITUATIONS

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## Abstract

This paper studies the nucleolus of graph-restricted games as an alternative for the Shapley value to evaluate communication situations. We focus on the inheritance of properties of cooperative games related to the nucleolus: balancedness (the nucleolus is in the core), compromise stability and strong compromise admissibility (these properties allow for a direct, closed formula for the nucleolus). We characterize the families of graphs for which the graph-restricted games inherit these properties from the underlying games. Moreover, for each of these properties, we characterize the family of graphs for which the nucleolus is invariant.

**Keywords:** nucleolus, communication situations, graph-restricted game, inheritance of properties, compromise stability, strong compromise admissibility, invariance

**JEL classification:** C71

## 1 Introduction

In a cooperative game with transferable utility, players can coordinate their actions and in particular obtain a joint monetary profit as a group. One of the main issues in cooperative game theory is the allocation of this joint profit among the players, taking into account the economic possibilities of all coalitions. Two distinguished solutions that solve this issue are the Shapley value (cf. Shapley, 1953) and the nucleolus (cf. Schmeidler, 1969).

Myerson (1977) extended cooperative games by introducing communication situations in which the communication restrictions of the players are modeled by a communication graph. The corresponding graph-restricted game is a modified cooperative game in which the communication restrictions are taken into account.

The Myerson value (cf. Myerson, 1977) of a communication situation is defined as the Shapley value of the corresponding graph-restricted game. This value is axiomatically characterized by Myerson (1980) and studied in several other contexts as well: hypergraphs (cf. Van den Nouweland, Borm, and Tijs, 1992), union stable structures (cf. Algaba, Bilbao, Borm, and López, 2001) and antimatroids (cf. Algaba, Bilbao, Van den Brink, and Jiménez-Losada, 2004).

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Moreover, several studies are devoted to the inheritance of properties of cooperative games that are related to the Shapley value. In particular, Owen (1986) studied the inheritance of superadditivity, Van den Nouweland and Borm (1991) studied convexity and Slikker (2000) studied, among others, average convexity.

Also the nucleolus is studied in the context of communication situations. Potters and Reijnierse (1995) showed that the nucleolus is the unique element of the kernel if the communication graph is a tree. Reijnierse and Potters (1998) and Katsev and Yanovskaya (2013) studied the collection of coalitions that determine the nucleolus and prenucleolus, respectively. Khmelnitskaya and Sudhölter (2013) provided an axiomatic characterization of the prenucleolus for games with communication structures. Instead, we follow the lines initiated by Owen (1986) and focus on the *inheritance* of several properties of cooperative games that are related to the nucleolus. Moreover, we study the *invariance* of the nucleolus, that is, the feature that the nucleolus of the graph-restricted game equals the nucleolus of the underlying game of a communication situation.

For the *inheritance*, we concentrate on properties of cooperative games that are related to the nucleolus from a computational point of view. In particular, we characterize families of graphs that guarantee the inheritance of balancedness, when the nucleolus is in the core, compromise stability and strong compromise admissibility. The last two properties allow for a direct, closed formula for the nucleolus. For every connected graph, the graph-restricted game satisfies balancedness for all communication situations with an underlying balanced game. To guarantee that compromise stability is inherited, the graph needs to be complete. The family of complete graphs is the largest family of graphs such that the graph-restricted game is compromise stable for all communication situations with an underlying compromise stable game. Finally, the family of biconnected graphs is the largest family of graphs to guarantee the inheritance of strong compromise admissibility.

For the *invariance* of the nucleolus, we identify families of graphs for which it is guaranteed that the nucleolus of the graph-restricted game equals the nucleolus of the underlying game for several classes of communication situations. We reconsider the classes of communication situations in which the underlying game satisfies balancedness, compromise stability and strong compromise admissibility. For balancedness and compromise stability, the graph needs to be complete in order to guarantee the invariance. For every connected graph that is not complete, we construct a communication situation with an underlying compromise stable game such that the nucleolus of the graph-restricted game is not equal to the nucleolus of the underlying game. Moreover, we show that biconnectedness is the weakest condition on the graph for which invariance of the nucleolus is guaranteed for all communication situations with an underlying strongly compromise admissible game. Interestingly, restricting to simple games, this result can be strengthened. Biconnectedness is also the weakest condition on the graph for which invariance of the nucleolus is guaranteed for all communication situations with an underlying compromise stable and simple game.

This paper is structured in the following way. Section 2 provides all relevant preliminaries on cooperative game theory and graph theory. Section 3 studies the inheritance of balancedness, compromise stability and strong compromise admissibility. Section 4 studies the invariance of the nucleolus.

## 2 Preliminaries

A (transferable utility) *cooperative game* is a pair  $(N, v)$  where  $N$  is a non-empty, finite set of *players* and  $v : 2^N \rightarrow \mathbb{R}$  a *characteristic function* with  $v(\emptyset) = 0$ . Here,  $2^N$  is the collection of all subsets (called *coalitions*) of  $N$  and  $v(S)$  is the *worth* of coalition  $S \in 2^N$ , representing the joint monetary rewards this coalition can obtain on its own. The class of all cooperative games with player set  $N$  is denoted by  $TU^N$ , and a cooperative game  $(N, v)$  is also denoted by  $v \in TU^N$ .

For a cooperative game  $v \in TU^N$ , the *imputation set* is given by

$$I(v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N \right\},$$

the *core* is given by

$$C(v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N \right\},$$

and the *core cover* (cf. Tijs and Lipperts, 1982) is given by

$$CC(v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N) \text{ and } m(v) \leq x \leq M(v) \right\},$$

where  $M(v), m(v) \in \mathbb{R}^N$  are, for all  $i \in N$ , defined by

$$M_i(v) = v(N) - v(N \setminus \{i\}),$$

and

$$m_i(v) = \max_{S \in 2^N : i \in S} \left\{ v(S) - \sum_{j \in S, j \neq i} M_j(v) \right\}.$$

A cooperative game  $v \in TU^N$  is called

- *imputation admissible* if  $I(v) \neq \emptyset$ ;
- *balanced* if  $C(v) \neq \emptyset$ ;
- *compromise stable* if  $CC(v) \neq \emptyset$  and  $C(v) = CC(v)$ , or equivalently if  $CC(v) \neq \emptyset$  and  $v(S) \leq \max \left\{ \sum_{i \in S} m_i(v), v(N) - \sum_{j \in N \setminus S} M_j(v) \right\}$  for all  $S \in 2^N \setminus \{\emptyset\}$  (cf. Quant, Borm, Reijnierse, and Van Velzen, 2005);
- *strongly compromise admissible* if  $CC(v) \neq \emptyset$  and  $v(S) \leq v(N) - \sum_{j \in N \setminus S} M_j(v)$  for all  $S \in 2^N \setminus \{\emptyset\}$  (cf. Driessen, 1988).

Note that strong compromise admissibility implies compromise stability, compromise stability implies balancedness, and balancedness implies imputation admissibility. Moreover, for a cooperative game with two players, all notions are equivalent. For a three player game, only balancedness and compromise stability are equivalent, while all notions differ for games with more than three players. Examples of compromise stable games are, among others, big boss

games (cf. Muto, Nakayama, Potters, and Tijs, 1988), clan games (cf. Potters, Poos, Tijs, and Muto, 1989) and bankruptcy games (cf. O'Neill, 1982 and Curiel, Maschler, and Tijs, 1987).

Let  $v \in TU^N$  be an imputation admissible game. The *excess* of a coalition  $S \in 2^N$  with respect to an imputation  $x \in I(v)$  is defined as  $Exc(S, x, v) = v(S) - \sum_{i \in S} x_i$ , while the *excess vector*  $\theta(x) \in \mathbb{R}^{2^N}$  is defined as the vector consisting of the excesses in non-increasing order, i.e.  $\theta(x)_k \geq \theta(x)_{k+1}$  for all  $k \in \{1, \dots, 2^N - 1\}$ . The *nucleolus* (cf. Schmeidler, 1969)  $nuc(v) \in \mathbb{R}^N$  is the unique imputation for which  $\theta(nuc(v)) \preceq \theta(x)$  for all  $x \in I(v)$ , where  $\preceq$  denotes the lexicographical order. It is known that  $nuc(v) \in C(v)$  for all balanced games  $v \in TU^N$ .

A collection  $B \subseteq 2^N \setminus \{\emptyset\}$  is called *balanced* if there exists a function  $\lambda : B \rightarrow \mathbb{R}_{++}$  such that  $\sum_{S \in B: i \in S} \lambda(S) = 1$  for all  $i \in N$ . According to the *Kohlberg criterion* (cf. Kohlberg, 1971), for a balanced game  $v \in TU^N$  and an imputation  $x \in I(v)$ , it holds that  $x = nuc(v)$  if and only if the collection  $\bigcup_{k=1}^s B_k(x, v)$  is balanced for all  $s \in \{1, \dots, t(x)\}$ , where  $B_k(x, v)$  is recursively defined by:

$$B_1(x, v) = \{S \in 2^N \setminus \{\emptyset, N\} \mid Exc(S, x, v) \geq Exc(T, x, v) \text{ for all } T \in 2^N \setminus \{\emptyset, N\}\},$$

and for all  $k \in \{2, \dots, t(x)\}$ :

$$B_k(x, v) = \left\{ S \in 2^N \setminus \{\emptyset, N\} \mid S \not\subseteq \bigcup_{r=1}^{k-1} B_r(x, v) \text{ and } Exc(S, x, v) \geq Exc(T, x, v) \right. \\ \left. \text{for all } T \in 2^N \setminus \{\emptyset, N\} \text{ with } T \not\subseteq \bigcup_{r=1}^{k-1} B_r(x, v) \right\}.$$

Here,  $t(x) \in \mathbb{N}$  is the unique number such that  $B_k(x, v) \neq \emptyset$  for all  $k \in \{1, \dots, t(x)\}$  and  $B_{t(x)+1}(x, v) = \emptyset$ .

A *bankruptcy problem* (cf. O'Neill, 1982) is a triple  $(N, A, c)$  where  $N$  is a non-empty, finite set of *players*,  $A \in \mathbb{R}_+$  and  $c \in \mathbb{R}_+^N$  consists of the *claims* of the players on  $A$  such that  $\sum_{i \in N} c_i \geq A$ . For a bankruptcy problem  $(N, A, c)$ , the *constrained equal awards rule* (CEA) allocates  $CEA_i(N, A, c) = \min\{\alpha, c_i\}$  for all  $i \in N$ , where  $\alpha \in \mathbb{R}$  is such that  $\sum_{i \in N} \min\{\alpha, c_i\} = A$ , while the *Talmud rule* (TAL) (cf. Aumann and Maschler, 1985) allocates

$$TAL(N, A, c) = \begin{cases} CEA(N, A, \frac{1}{2}c), & \text{if } \sum_{i \in N} c_i \geq 2A; \\ c - CEA(N, \sum_{i \in N} c_i - A, \frac{1}{2}c), & \text{if } \sum_{i \in N} c_i < 2A. \end{cases}$$

For compromise stable games and strongly compromise admissible games, the nucleolus can be described by a direct, closed formula.

**Proposition 2.1** [cf. Quant et al. (2005) and Driessen (1988)] *Let  $v \in TU^N$ .*

i) *If  $v$  is compromise stable, then, for all  $i \in N$ ,*

$$nuc_i(v) = m_i(v) + TAL_i \left( N, v(N) - \sum_{j \in N} m_j(v), M(v) - m(v) \right);$$

ii) *If  $v$  is strongly compromise admissible, then, for all  $i \in N$ ,*

$$nuc_i(v) = M_i(v) - \frac{1}{|N|} \left( \sum_{j \in N} M_j(v) - v(N) \right).$$

A *graph* is a pair  $(N, E)$ , where  $N$  is a non-empty, finite set of *players*, with  $|N| \geq 3$  and  $E \subseteq \{\{i, j\} \mid i, j \in N, i \neq j\}$  a finite set of *edges*. For a graph  $(N, E)$  and a subset of players  $S \in 2^N \setminus \{\emptyset\}$ , the *induced subgraph* on  $S$  is defined as the graph  $(S, E_S)$ , where  $E_S = \{\{i, j\} \in E \mid i, j \in S\}$ . A *path* in a graph  $(N, E)$  is defined as a sequence of players  $(i_0, \dots, i_m)$  such that  $i_k \neq i_\ell$  for all  $k, \ell \in \{0, 1, \dots, m\}, k \neq \ell$  and  $\{i_{k-1}, i_k\} \in E$  for all  $k \in \{1, \dots, m\}$ .

A graph  $(N, E)$  is called

- *connected* if for all  $i, j \in N$ , there is a path  $(i, \dots, j)$ ;
- *complete* if  $\{i, j\} \in E$  for all  $i, j \in N$ ;
- *biconnected* if for all  $i \in N$  the induced subgraph  $(N \setminus \{i\}, E_{N \setminus \{i\}})$  is connected;
- a *star* if there exists a player  $i \in N$  such that  $E = \{\{i, j\} \mid j \in N \setminus \{i\}\}$ .

Note that every complete graph is biconnected and that every biconnected graph is connected. Also a star is connected. For a graph  $(N, E)$ , a *component*  $C \in 2^N \setminus \{\emptyset\}$  is defined as a maximal (inclusion-wise) subset of players such that the induced subgraph  $(C, E_C)$  is connected. For a graph  $(N, E)$  and a subset of players  $S \in 2^N \setminus \{\emptyset\}$ , let  $S/E$  denote the set of all components in the induced subgraph  $(S, E_S)$ .

For a graph  $(N, E)$  and a cooperative game  $v \in TU^N$ , the *graph-restricted game*  $v^E \in TU^N$  is (cf. Myerson, 1977), for all  $S \in 2^N \setminus \{\emptyset\}$ , defined by

$$v^E(S) = \sum_{C \in S/E} v(C).$$

A *communication situation* (cf. Myerson, 1977) is a triple  $(N, v, E)$  where  $|N| \geq 3$ ,  $v \in TU^N$  and  $(N, E)$  a connected graph such that, for all  $S \in 2^N$ ,

$$v^E(S) \leq v(S).$$

Here, the graph is assumed to be connected. For, otherwise, one could deal with each connected component separately. Moreover, we assume that there are at least three players. If we would allow for only two players, any graph-restricted game would coincide with the underlying cooperative game. We do not assume that the underlying game is zero-normalized; also individual players could obtain a monetary value. Finally, to adequately reflect the communication restrictions for cooperation among the players, we assume that the worth of a coalition in the graph-restricted game is at most the worth of this coalition in the underlying cooperative game. The following lemma shows that the latter assumption is satisfied if the underlying game is superadditive. A cooperative game  $v \in TU^N$  is called *superadditive* if  $v(S) + v(T) \leq v(S \cup T)$  for all  $S, T \in 2^N$  for which  $S \cap T = \emptyset$ .

**Lemma 2.1** *Let  $(N, E)$  be a graph and let  $v \in TU^N$  be superadditive. Then,  $v^E(S) \leq v(S)$  for all  $S \in 2^N \setminus \{\emptyset\}$ .*

*Proof:* Let  $S \in 2^N \setminus \{\emptyset\}$ . Since  $C \cap C' = \emptyset$  for any two components  $C, C' \in S/E$ , it follows that

$$v^E(S) = \sum_{C \in S/E} v(C) \leq v \left( \bigcup_{C \in S/E} C \right) = v(S),$$

where the inequality follows from (repeatedly) applying superadditivity. □

Given a connected graph  $(N, E)$ , we denote the class of all communication situations by  $TU^{N,E}$ . With a slight abuse of notation, a communication situation is denoted by  $v \in TU^{N,E}$ . Furthermore, we denote the subclass with an underlying balanced game by  $BAL^{N,E}$ , the subclass with an underlying compromise stable game by  $CS^{N,E}$ , and the subclass with an underlying strongly compromise admissible game by  $SCA^{N,E}$ .

### 3 Inheritance of properties

This section studies the inheritance of balancedness, compromise stability and strong compromise admissibility. For each of these properties, we characterize the largest family of (connected) graphs for which the graph-restricted game satisfies this property for all communication situations with an underlying game satisfying this property. First, we show that balancedness is always inherited.<sup>1</sup> This was also remarked by Van den Nouweland and Borm (1991).

**Theorem 3.1** *Let  $(N, E)$  be a connected graph. Then,  $v^E$  is balanced for all  $v \in BAL^{N,E}$ .*

*Proof:* Let  $v \in BAL^{N,E}$  and let  $x \in C(v)$ . Then,  $\sum_{i \in N} x_i = v(N) = v^E(N)$  and  $\sum_{i \in S} x_i \geq v(S) \geq v^E(S)$  for all  $S \in 2^N$ . Hence,  $x \in C(v^E)$ .  $\square$

Since balancedness and compromise stability are equivalent for a three player game, compromise stability is always inherited for communication situations with three players. For more than three players, compromise stability implies balancedness and therefore, the graph-restricted game is balanced for all communication situations with an underlying compromise stable game. However, to guarantee the inheritance of compromise stability itself, the graph needs to be complete. We show that for every connected graph with more than three players that is not complete, one can find a communication situation with an underlying compromise stable game such that the graph-restricted game is not compromise stable. For this, we need the following lemma.

**Lemma 3.1** *Let  $(N, E)$  be a connected graph. Then,  $M(v^E) \geq M(v)$  and  $m(v^E) \leq m(v)$  for all  $v \in TU^{N,E}$ . Consequently,  $CC(v) \subseteq CC(v^E)$ .*

*Proof:* Let  $v \in TU^{N,E}$ . Since  $v^E(N) = v(N)$  and  $v^E(S) \leq v(S)$  for all  $S \in 2^N$ , we have

$$M_i(v^E) = v^E(N) - v^E(N \setminus \{i\}) \geq v(N) - v(N \setminus \{i\}) = M_i(v),$$

for all  $i \in N$ . Using this, we have that

$$m_i(v^E) = \max_{S \in 2^N: i \in S} \left\{ v^E(S) - \sum_{j \in S, j \neq i} M_j(v^E) \right\} \leq \max_{S \in 2^N: i \in S} \left\{ v(S) - \sum_{j \in S, j \neq i} M_j(v) \right\} = m_i(v),$$

for all  $i \in N$ . Finally, for  $x \in CC(v)$ , we have

$$\sum_{i \in N} x_i = v(N) = v^E(N),$$

and

$$m(v^E) \leq m(v) \leq x \leq M(v) \leq M(v^E).$$

Consequently,  $x \in CC(v^E)$ .  $\square$

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<sup>1</sup>In fact, it can be shown that any prosperity property (cf. Gellekom, Potters, and Reijnierse, 1999) is always inherited. Also additivity is always inherited.



**Theorem 3.2** *Let  $(N, E)$  be a connected graph. Then,  $v^E$  is compromise stable for all  $v \in CS^{N,E}$  if and only if  $|N| = 3$  or  $(N, E)$  is complete.*

*Proof:* First consider the ‘if’ part: if  $|N| = 3$ , then compromise stability is equivalent to balancedness and Theorem 3.1 implies that  $v^E$  is compromise stable for all  $v \in CS^{N,E}$ . If  $(N, E)$  is complete, then  $v^E = v$  and  $v^E$  is compromise stable for all  $v \in CS^{N,E}$ .

Next, the ‘only if’ part: let  $v^E$  be compromise stable for all  $v \in CS^{N,E}$ . We prove that, if  $|N| > 3$  and  $(N, E)$  is not complete, we can construct a communication situation with an underlying compromise stable game for which the graph-restricted game is not compromise stable. Let  $|N| > 3$  and let  $(N, E)$  be not complete. Set  $N = \{1, 2, 3, 4, \dots, n\}$  and w.l.o.g. assume that  $\{1, 2\} \notin E$ , while  $\{1, 3\} \in E$ . Consider  $v \in CS^{N,E}$  with, for all  $S \in 2^N \setminus \{\emptyset\}$ ,

$$v(S) = \begin{cases} 7, & \text{if } S = N; \\ 6, & \text{if } S = N \setminus \{1\}; \\ 5, & \text{if } S = N \setminus \{2\}; \\ 4, & \text{if } S = N \setminus \{4\}; \\ 3, & \text{if } S \notin \{N, N \setminus \{1\}, N \setminus \{2\}, N \setminus \{4\}\}, \text{ and } \{1, 2\} \subseteq S \text{ or } \{1, 3\} \subseteq S; \\ 0, & \text{otherwise.} \end{cases}$$

Note that  $M(v) = (1, 2, 4, 3, 4, \dots, 4)$  and  $m(v) = (1, 2, 2, 0, 0, \dots, 0)$ , which means that  $CC(v) \neq \emptyset$ . Obviously, the inequality  $v(S) \leq \max \left\{ \sum_{i \in S} m_i(v), v(N) - \sum_{j \in N \setminus S} M_j(v) \right\}$  is satisfied for  $S \in \{N, N \setminus \{1\}, N \setminus \{2\}, N \setminus \{4\}\}$  and for all  $S \in 2^N$  with  $v(S) = 0$ . For  $S \in 2^N, S \notin \{N, N \setminus \{1\}, N \setminus \{2\}, N \setminus \{4\}\}$ , it holds that  $v(S) \leq m_1(v) + m_2(v)$  if  $\{1, 2\} \subseteq S$ , and  $v(S) \leq m_1(v) + m_3(v)$  if  $\{1, 3\} \subseteq S$ . Hence,  $v$  is compromise stable.

We show that  $v^E$  is not compromise stable, by showing that

$$v^E(\{1, 3\}) > \max \left\{ m_1(v^E) + m_3(v^E), v^E(N) - \sum_{j \in N, j \neq 1, 3} M_j(v^E) \right\}.$$

First, note that  $v^E(\{1, 3\}) = v(\{1, 3\}) = 3$ . Secondly, with regard to  $m_1(v)$ , we have that  $v(S) - \sum_{j \in S, j \neq 1} M_j(v) \leq 0$  for all  $S \in 2^N, S \neq \{1, 2\}$  and  $v(\{1, 2\}) - M_2(v) = 1$ . Using the fact that  $v^E(S) \leq v(S)$  for all  $S \in 2^N$  and  $M(v^E) \geq M(v)$  (according to Lemma 3.1), it can be seen that  $v^E(S) - \sum_{j \in S, j \neq 1} M_j(v^E) \leq 0$  for all  $S \in 2^N, S \neq \{1, 2\}$ . Moreover,  $v^E(\{1, 2\}) = 0$  (since  $\{1, 2\} \notin E$ ) and hence,  $v^E(\{1, 2\}) - M_2(v^E) \leq 0$ . Consequently,  $m_1(v^E) = 0$ . Lemma 3.1 also implies that  $m_3(v^E) \leq m_3(v) = 2$ . Hence,

$$m_1(v^E) + m_3(v^E) \leq 2.$$

Finally,  $v^E(N) = v(N) = 7$ ,  $M_2(v^E) \geq M_2(v) = 2$ ,  $M_4(v^E) \geq M_4(v) = 3$  and  $M_j(v^E) \geq M_j(v) \geq 0$  for all  $j \in N$ , imply that

$$\begin{aligned} v^E(N) - \sum_{\substack{j \in N, \\ j \neq 1, 3}} M_j(v^E) &= v^E(N) - M_2(v^E) - M_4(v^E) - \sum_{\substack{j \in N, \\ j \neq 1, 2, 3, 4}} M_j(v^E) \\ &\leq 7 - 2 - 3 = 2. \end{aligned}$$

Subsequently,  $v^E$  is not compromise stable. □

Theorem 3.2 characterizes the family of graphs for which the graph-restricted games inherit compromise stability from the underlying games. Lemma 3.2 shows that it is not possible to guarantee strong compromise admissibility for the graph-restricted games if the underlying games satisfy compromise stability. Moreover, if there are at least four players, it is also not possible to guarantee compromise stability for the graph-restricted games if the underlying games are balanced.

**Lemma 3.2** *Let  $(N, E)$  be a connected graph. Then the following two statements hold:*

- i) *If  $|N| \geq 4$ , then there exists a  $v \in \text{BAL}^{N,E}$  such that  $v^E$  is not compromise stable;*
- ii) *There exists a  $v \in \text{CS}^{N,E}$  such that  $v^E$  is not strongly compromise admissible.*

*Proof:* The first statement is a direct consequence of the proof of Theorem 3.2. For the second statement, set  $N = \{1, 2, \dots, n\}$  and w.l.o.g. let  $\{1, 2\} \in E$ . Consider the communication situation  $v \in \text{CS}^{N,E}$  with, for all  $S \in 2^N \setminus \{\emptyset\}$ ,

$$v(S) = \begin{cases} 1, & \text{if } \{1, 2\} \subseteq S; \\ 0, & \text{otherwise.} \end{cases}$$

Note that  $M(v) = (1, 1, 0, \dots, 0)$  and  $m(v) = (0, 0, 0, \dots, 0)$ , which means that  $CC(v) \neq \emptyset$ . Moreover,  $v(S) \leq v(N) - \sum_{j \in N \setminus S} M_j(v)$  for all  $S \in 2^N$  for which  $\{1, 2\} \subseteq S$  and  $v(S) \leq \sum_{i \in S} m_i(v)$  for all other  $S \in 2^N$ . Hence,  $v$  is compromise stable.

Furthermore,  $v^E = v$ , which is not strongly compromise admissible, because

$$v(\{3\}) = 0 > -1 = v(N) - \sum_{j \in N \setminus S} M_j(v). \quad \square$$

A specific subclass of compromise stable games is the class of strongly compromise admissible games. The graph-restricted game corresponding to a communication situation with an underlying strongly compromise admissible game is balanced for any connected graph, since strong compromise admissibility implies balancedness and balancedness is always inherited. We first show that the graph-restricted game is strongly compromise admissible for any communication situation with an underlying strongly compromise admissible game if the graph is biconnected.

**Lemma 3.3** *Let  $(N, E)$  be a connected graph. If  $(N, E)$  is biconnected, then  $v^E$  is strongly compromise admissible for all  $v \in \text{SCA}^{N,E}$ .*

*Proof:* Let  $(N, E)$  be biconnected and let  $v \in \text{SCA}^{N,E}$ . First,  $CC(v) \neq \emptyset$  implies  $CC(v^E) \neq \emptyset$  by using Lemma 3.1. Secondly, since  $(N, E)$  is biconnected,  $M_i(v^E) = M_i(v)$  for all  $i \in N$  and hence, for all  $S \in 2^N \setminus \{\emptyset\}$ ,

$$v^E(S) \leq v(S) \leq v(N) - \sum_{j \in N \setminus S} M_j(v) = v^E(N) - \sum_{j \in N \setminus S} M_j(v^E).$$

Consequently,  $v^E$  is strongly compromise admissible.  $\square$

Next, we characterize the family of graphs for which the graph-restricted game satisfies compromise stability for all communication situations with an underlying strongly compromise admissible game. It turns out that this family of graphs consists of all biconnected graphs and stars.

**Theorem 3.3** *Let  $(N, E)$  be a connected graph. Then,  $v^E$  is compromise stable for all  $v \in \text{SCA}^{N,E}$  if and only if  $(N, E)$  is biconnected or a star.*

*Proof:* First consider the ‘if’ part: if  $(N, E)$  is biconnected, then the statement follows from Lemma 3.3, since strong compromise admissibility implies compromise stability. If the graph is a star, then let  $k \in N$  such that  $E = \{\{i, k\} \mid i \in N \setminus \{k\}\}$ . Let  $v \in SCA^{N,E}$ . Note that, by Lemma 3.1,  $CC(v^E) \neq \emptyset$ . Moreover, let  $S \in 2^N \setminus \{\emptyset\}$ . If  $k \notin S$ , then

$$v^E(S) = \sum_{i \in S} v(\{i\}) = \sum_{i \in S} v^E(\{i\}) \leq \sum_{i \in S} m_i(v^E).$$

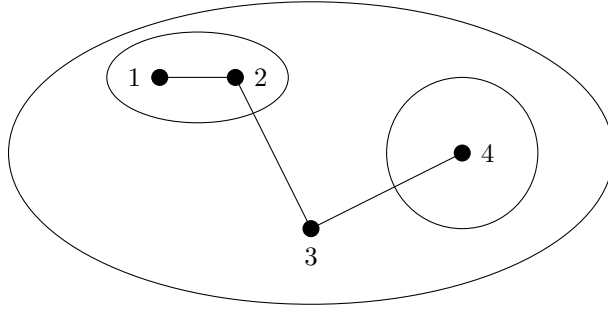
If  $k \in S$ , then  $M_j(v^E) = M_j(v)$  for all  $j \in N \setminus S$  and hence,

$$v^E(S) \leq v(S) \leq v(N) - \sum_{j \in N \setminus S} M_j(v) = v^E(N) - \sum_{j \in N \setminus S} M_j(v^E).$$

Consequently,  $v^E$  is compromise stable.

Next, the ‘only if’ part: let  $v^E$  be compromise stable for all  $v \in SCA^{N,E}$ . We prove that, if  $(N, E)$  is not biconnected and not a star, we can construct a communication situation with an underlying strongly compromise admissible game for which the graph-restricted game is not compromise stable. Let  $(N, E)$  be not biconnected and not a star. Then  $|N| \geq 4$ . Set  $N = \{1, 2, 3, 4, \dots, n\}$  and w.l.o.g. assume that  $\{1, 2\}, \{2, 3\}, \{3, 4\} \in E$ , the induced subgraph on  $N \setminus \{3\}$  is not connected, and that players 1 and 2 are in one component of the induced subgraph on  $N \setminus \{3\}$  and player 4 is in another (Figure 1 provides a schematic representation). Consider  $v \in SCA^{N,E}$  with, for all  $S \in 2^N \setminus \{\emptyset\}$ ,

$$v(S) = \begin{cases} 8, & \text{if } S = N; \\ 8, & \text{if } S = N \setminus \{j\} \text{ for } j \in N \setminus \{1, 2, 3, 4\}; \\ 6, & \text{if } S \in \{N \setminus \{1\}, N \setminus \{2\}, N \setminus \{3\}, N \setminus \{4\}\}; \\ 3, & \text{if } |S| \leq n - 2 \text{ and } \{1, 2\} \subseteq S; \\ 0, & \text{otherwise.} \end{cases}$$



**Figure 1** – Schematic representation of the graph  $(N, E)$

It can be readily checked that  $M(v) = m(v) = (2, 2, 2, 2, 0, \dots, 0)$ . Hence,  $CC(v) \neq \emptyset$ . Obviously,  $v(S) \leq v(N) - \sum_{j \in N \setminus S} M_j(v)$  holds for  $S \in 2^N$  for which  $|S| > n - 2$ . For  $S \in 2^N$  for which  $|S| \leq n - 2$  and  $\{1, 2\} \subseteq S$ , we have  $v(N) - \sum_{j \in N \setminus S} M_j(v) \geq 8 - 4 \geq 3 = v(S)$ . Finally, for  $S \in 2^N$  for which  $v(S) = 0$ ,  $v(N) - \sum_{j \in N \setminus S} M_j(v) \geq 8 - 8 = 0$ . Hence,  $v$  is strongly compromise admissible.

We show that  $v^E$  is not compromise stable, by showing that

$$v^E(\{1, 2\}) > \max \left\{ m_1(v^E) + m_2(v^E), v^E(N) - \sum_{j \in N, j \neq 1, 2} M_j(v^E) \right\}.$$

First, note that  $v^E(\{1, 2\}) = v(\{1, 2\}) = 3$ . Secondly, since  $v^E(N \setminus \{3\}) = v(\{1, 2\}) = 3$  (due to the fact that the induced subgraph on  $N \setminus \{3\}$  is not connected, but consists of at least one component with  $\{1, 2\} \in E$ ), we have that  $M_3(v^E) = 5$ . Using Lemma 3.1, we have that  $M(v^E) \geq M(v) \geq 0$  and in particular,  $M_4(v^E) \geq M_4(v) = 2$ , such that it follows that

$$v^E(N) - \sum_{j \in N, j \neq 1, 2} M_j(v^E) \leq v^E(N) - M_3(v^E) - M_4(v^E) \leq 8 - 5 - 2 = 1.$$

Moreover, we claim that  $m_1(v^E) = \max_{S \in 2^N: 1 \in S} \left\{ v^E(S) - \sum_{j \in S, j \neq 1} M_j(v^E) \right\} \leq 1$ : for  $S = N$  and  $S = N \setminus \{j\}$  for  $j \in N \setminus \{1, 2, 3\}$ , we see that  $\{2, 3\} \subseteq S$  and  $v^E(S) \leq 8$ , and, consequently,

$$v^E(S) - \sum_{j \in S, j \neq 1} M_j(v^E) \leq v^E(S) - M_2(v^E) - M_3(v^E) \leq 8 - 2 - 5 = 1.$$

For  $S = N \setminus \{3\}$ , we have that  $v^E(S) = 3$  and  $M_2(v^E) \geq 2$ , and, consequently,

$$v^E(S) - \sum_{j \in S, j \neq 1} M_j(v^E) \leq v^E(S) - M_2(v^E) \leq 3 - 2 = 1.$$

For  $S = N \setminus \{2\}$ , we have that  $\{3, 4\} \subseteq S$  and  $v^E(S) \leq 6$ , and, consequently,

$$v^E(S) - \sum_{j \in S, j \neq 1} M_j(v^E) \leq v^E(S) - M_3(v^E) - M_4(v^E) \leq 6 - 5 - 2 = -1.$$

For all  $S \in 2^N$  with  $|S| \leq n - 2$  and  $\{1, 2\} \subseteq S$ , we have that  $v^E(S) \leq v(S) = 3$  and  $2 \in S$ , and, consequently,

$$v^E(S) - \sum_{j \in S, j \neq 1} M_j(v^E) \leq v(S) - M_2(v^E) \leq 3 - 2 = 1.$$

Finally, for  $S \in 2^N$  with  $v(S) = 0$  it clearly holds that  $v^E(S) - \sum_{j \in S, j \neq 1} M_j(v^E) \leq 0$ .

We may conclude that  $m_1(v^E) \leq 1$ . Similarly, one can show that  $m_2(v^E) \leq 1$  and thus

$$m_1(v^E) + m_2(v^E) \leq 2.$$

Consequently,  $v^E$  is not compromise stable. □

Finally, the family of biconnected graphs is the largest family of (connected) graphs for which the graph-restricted game is strongly compromise admissible for all communication situations with an underlying strongly compromise admissible game, as the following theorem shows.

**Theorem 3.4** *Let  $(N, E)$  be a connected graph. Then,  $v^E$  is strongly compromise admissible for all  $v \in SCA^{N, E}$  if and only if  $(N, E)$  is biconnected.*

*Proof:* First consider the ‘if’ part: if  $(N, E)$  is biconnected, then  $v^E$  is strongly compromise admissible for all  $v \in SCA^{N, E}$  according to Lemma 3.3.

Next, the ‘only if’ part: let  $v^E$  is strongly compromise admissible for all  $v \in SCA^{N, E}$ . We prove that, if  $(N, E)$  is not biconnected, we can construct a communication situation with an underlying strongly compromise admissible game for which the graph-restricted game is not strongly compromise admissible. Let  $(N, E)$  be not biconnected. Set  $N = \{1, 2, \dots, n\}$  and w.l.o.g. we can assume that  $\{1, 2\}, \{2, 3\} \in E$ , the induced subgraph on  $N \setminus \{2\}$  is not connected

and that players 1 and 3 are in two different components in the induced subgraph on  $N \setminus \{2\}$ . Consider  $v \in SCA^{N,E}$  with, for all  $S \in 2^N \setminus \{\emptyset\}$ ,

$$v(S) = \begin{cases} 1, & \text{if } \{1, 2\} \subseteq S \text{ or } \{1, 3\} \subseteq S; \\ 0, & \text{otherwise.} \end{cases}$$

Note that  $CC(v) \neq \emptyset$ , since it can be readily checked that  $M(v) = m(v) = (1, 0, 0, \dots, 0)$ . Moreover, for  $S \in 2^N$  for which  $1 \in S$ ,  $v(S) \leq 1 = v(N) - \sum_{j \in N \setminus S} M_j(v)$ . For  $S \in 2^N$  for which  $1 \notin S$ ,  $v(S) = 0 = v(N) - \sum_{j \in N \setminus S} M_j(v)$ . Hence,  $v$  is strongly compromise admissible.

We show that  $v^E$  is not strongly compromise admissible, by showing that

$$v^E(\{3\}) > v^E(N) - \sum_{j \in N, j \neq 3} M_j(v^E).$$

First, note that  $v^E(\{3\}) = v(\{3\}) = 0$ . Secondly, since  $v^E(N \setminus \{2\}) = 0$  (due to the fact that players 1 and 3 are in two different components of the induced subgraph on  $N \setminus \{2\}$ ), we have that  $M_2(v^E) = 1$ . Using Lemma 3.1,  $M(v^E) \geq M(v) \geq 0$  and in particular,  $M_1(v^E) \geq 1$ . Hence,

$$v^E(N) - \sum_{j \in N, j \neq 3} M_j(v^E) \leq v^E(N) - M_1(v^E) - M_2(v^E) \leq -1.$$

Consequently,  $v^E$  is not strongly compromise admissible.  $\square$

To conclude this section, Table 1 provides a summary of the main results regarding the inheritance of properties. For each of the three properties, it identifies the largest family of graphs for which the graph-restricted game satisfies this property for all communication situations with an underlying game satisfying this property. In Table 1, the rows indicate communication situations with an arbitrary number of players and an underlying game satisfying a particular property, while the columns indicate the corresponding graph-restricted games.

		Graph-restricted games		
		Balancedness	Compromise stability	Strong compromise admissibility
Underlying games	Balancedness	all graphs	no graphs	no graphs
	Compromise stability	all graphs	complete graphs	no graphs
	Strong compromise admissibility	all graphs	biconnected graphs and stars	biconnected graphs

**Table 1** – *Survey of inheritance of properties*

## 4 Invariance of the nucleolus

In this section, we study the invariance of the nucleolus. That is, we characterize families of graphs for which the nucleolus of the graph-restricted game equals the nucleolus of the underlying game for several classes of communication situations. In particular, we reconsider the

classes of communication situations with respectively an underlying strongly compromise admissible game, an underlying compromise stable game and an underlying balanced game. Note that the graph-restricted game is imputation admissible if the underlying game is imputation admissible.

We start out with the class of communication situations in which the underlying games satisfy strong compromise admissibility.

**Theorem 4.1** *Let  $(N, E)$  be a connected graph. Then,  $\text{nuc}(v^E) = \text{nuc}(v)$  for all  $v \in \text{SCA}^{N,E}$  if and only if  $(N, E)$  is biconnected.*

*Proof:* First consider the ‘if’ part: if  $(N, E)$  is biconnected, then  $v^E$  is strongly compromise admissible for all  $v \in \text{SCA}^{N,E}$ , according to Theorem 3.4. Then, using Proposition 2.1, for  $v \in \text{SCA}^{N,E}$  and  $i \in N$ ,

$$\begin{aligned} \text{nuc}_i(v^E) &= M_i(v^E) - \frac{1}{|N|} \left( \sum_{j \in N} M_j(v^E) - v^E(N) \right) \\ &= M_i(v) - \frac{1}{|N|} \left( \sum_{j \in N} M_j(v) - v(N) \right) \\ &= \text{nuc}_i(v), \end{aligned}$$

since  $M(v^E) = M(v)$  for any biconnected graph  $(N, E)$ .

Next, the ‘only if’ part: let  $\text{nuc}(v^E) = \text{nuc}(v)$  for all  $v \in \text{SCA}^{N,E}$ . We prove that, if  $(N, E)$  is not biconnected, we can construct a communication situation  $v \in \text{SCA}^{N,E}$  such that  $\text{nuc}(v^E) \neq \text{nuc}(v)$ . Let  $(N, E)$  be not biconnected. Set  $N = \{1, 2, \dots, n\}$  and w.l.o.g. we can assume that  $\{1, 2\}, \{2, 3\} \in E$ , the induced subgraph on  $N \setminus \{2\}$  is not connected and that players 1 and 3 are in two different components in the induced subgraph on  $N \setminus \{2\}$ . Reconsider  $v \in \text{SCA}^{N,E}$  with, for all  $S \in 2^N \setminus \{\emptyset\}$ ,

$$v(S) = \begin{cases} 1, & \text{if } \{1, 2\} \subseteq S \text{ or } \{1, 3\} \subseteq S; \\ 0, & \text{otherwise.} \end{cases}$$

Since  $M(v) = (1, 0, 0, \dots, 0)$ , we have that, using Proposition 2.1,

$$\text{nuc}(v) = (1, 0, 0, \dots, 0).$$

Moreover, for all  $S \in 2^N \setminus \{\emptyset\}$ ,

$$v^E(S) = \begin{cases} 1, & \text{if } \{1, 2\} \subseteq S; \\ 0, & \text{otherwise,} \end{cases}$$

and consequently,

$$\text{nuc}(v^E) = \left(\frac{1}{2}, \frac{1}{2}, 0, \dots, 0\right).$$

Hence,  $\text{nuc}(v^E) \neq \text{nuc}(v)$ . □

Next, we reconsider the class of communication situations with an underlying compromise stable game. For this larger class, biconnected graphs are not sufficient to guarantee the invariance of the nucleolus. In fact, the weakest condition on the graph for which invariance of the nucleolus

is guaranteed for all communication situations with an underlying compromise stable game is completeness. The proof involves an intricate construction of a communication situation for every connected graph that is not complete, for which it holds that the nucleolus of the graph-restricted game is not equal to the nucleolus of the underlying game.

**Theorem 4.2** *Let  $(N, E)$  be a connected graph. Then,  $\text{nuc}(v^E) = \text{nuc}(v)$  for all  $v \in CS^{N,E}$  if and only if  $(N, E)$  is complete.*

*Proof:* First consider the ‘if’ part: if  $(N, E)$  is complete, then  $v^E = v$  and  $\text{nuc}(v^E) = \text{nuc}(v)$  for all  $v \in SC^{N,E}$ .

Next, the ‘only if’ part: let  $\text{nuc}(v^E) = \text{nuc}(v)$  for all  $v \in CS^{N,E}$ . We prove that, if  $(N, E)$  is not complete, we can construct a communication situation  $v \in CS^{N,E}$  such that  $\text{nuc}(v^E) \neq \text{nuc}(v)$ . Let  $(N, E)$  be not complete. We distinguish between two cases: either  $|N| = 3$  or  $|N| \geq 4$ . First, suppose that  $|N| = 3$  and set  $N = \{1, 2, 3\}$ . Assume w.l.o.g. that  $\{1, 3\} \notin E$ . Then,  $\{1, 2\}, \{2, 3\} \in E$ , since  $(N, E)$  is connected. Reconsider  $v \in CS^{N,E}$  with, for all  $S \in 2^N \setminus \{\emptyset\}$ ,

$$v(S) = \begin{cases} 1, & \text{if } \{1, 2\} \subseteq S \text{ or } \{1, 3\} \subseteq S; \\ 0, & \text{otherwise.} \end{cases}$$

For all  $S \in 2^N \setminus \{\emptyset\}$ ,

$$v^E(S) = \begin{cases} 1, & \text{if } \{1, 2\} \subseteq S; \\ 0, & \text{otherwise.} \end{cases}$$

Consequently,  $\text{nuc}(v^E) = (\frac{1}{2}, \frac{1}{2}, 0) \neq (1, 0, 0) = \text{nuc}(v)$ .

Secondly, suppose that  $|N| \geq 4$ . Set  $N = \{1, 2, 3, 4, \dots, n\}$  and assume w.l.o.g. that  $\{1, 2\} \notin E$  and  $\{1, 3\} \in E$ . Reconsider  $v \in CS^{N,E}$  with, for all  $S \in 2^N \setminus \{\emptyset\}$ ,

$$v(S) = \begin{cases} 7, & \text{if } S = N; \\ 6, & \text{if } S = N \setminus \{1\}; \\ 5, & \text{if } S = N \setminus \{2\}; \\ 4, & \text{if } S = N \setminus \{4\}; \\ 3, & \text{if } S \notin \{N, N \setminus \{1\}, N \setminus \{2\}, N \setminus \{4\}\}, \text{ and } \{1, 2\} \subseteq S \text{ or } \{1, 3\} \subseteq S; \\ 0, & \text{otherwise.} \end{cases}$$

Then,  $M(v) = (1, 2, 4, 3, 4, \dots, 4)$  and  $m(v) = (1, 2, 2, 0, 0, \dots, 0)$ , and using Proposition 2.1,

$$\text{nuc}_i(v) = \begin{cases} 1, & \text{if } i = 1; \\ 2, & \text{if } i = 2; \\ 2 + \frac{2}{n-2}, & \text{if } i = 3; \\ \frac{2}{n-2}, & \text{otherwise.} \end{cases}$$

To show that  $\text{nuc}(v^E) \neq \text{nuc}(v)$ , we use the Kohlberg criterion and show that  $B_1(\text{nuc}(v), v^E)$  is not balanced. For this, we need to identify the coalitions with the highest excess. Since

$$v^E(S) = \begin{cases} 0, & \text{if } S = \{j\} \text{ for } j \in N; \\ 0, & \text{if } S = \{1, 2\}; \\ 3, & \text{if } S = \{1, 3\}; \\ 0, & \text{if } S = \{2, 3\}, \end{cases}$$

one readily checks that

$$Exc(S, \text{nuc}(v), v^E) = \begin{cases} -1, & \text{if } S = \{1\}; \\ -2, & \text{if } S = \{2\}; \\ \frac{-2}{n-2} - 2, & \text{if } S = \{3\}; \\ \frac{-2}{n-2}, & \text{if } S = \{j\} \text{ for } j \in N \setminus \{1, 2, 3\}; \\ -3, & \text{if } S = \{1, 2\}; \\ \frac{-2}{n-2}, & \text{if } S = \{1, 3\}; \\ \frac{-2}{n-2} - 4, & \text{if } S = \{2, 3\}. \end{cases}$$

For  $S \in 2^N \setminus \{\emptyset, N\}$  for which  $S = \{1, 2, j\}$  for  $j \in N \setminus \{1, 2, 3\}$ , it holds that

$$v^E(S) = \begin{cases} 3, & \text{if the induced subgraph on } S \text{ is connected;} \\ 0, & \text{otherwise,} \end{cases}$$

and hence,

$$Exc(S, \text{nuc}(v), v^E) = \begin{cases} \frac{-2}{n-2}, & \text{if the induced subgraph on } S \text{ is connected;} \\ \frac{-2}{n-2} - 3, & \text{otherwise.} \end{cases}$$

Note that the induced subgraph on  $S$  is connected if and only if  $\{1, j\}, \{2, j\} \in E$ .

Furthermore, for  $S \in 2^N \setminus \{\emptyset, N\}$  for which  $3 < |S| < n - 1$ ,  $\{1, 2, j\} \subseteq S$  for  $j \in N \setminus \{1, 2, 3\}$  and  $3 \notin S$ , it holds that

$$v^E(S) \leq 3,$$

and hence,

$$Exc(S, \text{nuc}(v), v^E) < v^E(S) - \text{nuc}_1(v) - \text{nuc}_2(v) - \text{nuc}_j(v) \leq 3 - 1 - 2 - \frac{2}{n-2} = \frac{-2}{n-2},$$

since  $\text{nuc}_i(v) > 0$  for all  $i \in N$ . Subsequently, these coalitions can not be coalitions with the highest excess.

In addition, for  $S \in 2^N \setminus \{\emptyset, N\}$  for which  $2 < |S| < n - 1$  and  $\{1, 3\} \subseteq S$ , it holds that  $v^E(S) = 3$  and hence,

$$Exc(S, \text{nuc}(v), v^E) < v^E(S) - \text{nuc}_1(v) - \text{nuc}_3(v) = 3 - 1 - 2 - \frac{2}{n-2} = \frac{-2}{n-2}.$$

For  $S \in 2^N \setminus \{\emptyset, N\}$  for which  $1 < |S| < n - 1$ ,  $\{1, 2\} \not\subseteq S$ ,  $\{1, 3\} \not\subseteq S$  and  $j \in S$  for  $j \in N \setminus \{1, 2, 3\}$ , it holds that  $v^E(S) = 0$  and hence,

$$Exc(S, \text{nuc}(v), v^E) < v^E(S) - \text{nuc}_j(v) = 0 - \frac{2}{n-2} = \frac{-2}{n-2}.$$

For  $S \in 2^N \setminus \{\emptyset, N\}$  for which  $S = N \setminus \{j\}$  for  $j \in N \setminus \{1, 2, 3, 4\}$ , it holds that  $v^E(S) = 3$ , since  $\{1, 3\} \subseteq S$  and hence,

$$Exc(S, \text{nuc}(v), v^E) < v^E(S) - \text{nuc}_1(v) - \text{nuc}_3(v) = 3 - 1 - 2 - \frac{2}{n-2} = \frac{-2}{n-2}.$$



Finally, for the coalitions  $N \setminus \{4\}$ ,  $N \setminus \{3\}$ ,  $N \setminus \{2\}$  and  $N \setminus \{1\}$ , the worth of the coalition in the graph-restricted game depends on whether the induced subgraph is connected or not:

$$\begin{aligned} v^E(N \setminus \{4\}) &= \begin{cases} 4, & \text{if the induced subgraph on } N \setminus \{4\} \text{ is connected;} \\ 3, & \text{otherwise,} \end{cases} \\ v^E(N \setminus \{3\}) &= \begin{cases} 3, & \text{if the induced subgraph on } N \setminus \{3\} \text{ is connected;} \\ 0, & \text{otherwise,} \end{cases} \\ v^E(N \setminus \{2\}) &= \begin{cases} 5, & \text{if the induced subgraph on } N \setminus \{2\} \text{ is connected;} \\ 3, & \text{otherwise,} \end{cases} \\ v^E(N \setminus \{1\}) &= \begin{cases} 6, & \text{if the induced subgraph on } N \setminus \{1\} \text{ is connected;} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Consequently,

$$\begin{aligned} Exc(N \setminus \{4\}, \text{nuc}(v), v^E) &= \begin{cases} \frac{2}{n-2} - 3, & \text{if the induced subgraph on } N \setminus \{4\} \text{ is connected;} \\ \frac{2}{n-2} - 4, & \text{otherwise,} \end{cases} \\ Exc(N \setminus \{3\}, \text{nuc}(v), v^E) &= \begin{cases} \frac{2}{n-2} - 2, & \text{if the induced subgraph on } N \setminus \{3\} \text{ is connected;} \\ \frac{2}{n-2} - 5, & \text{otherwise,} \end{cases} \\ Exc(N \setminus \{2\}, \text{nuc}(v), v^E) &= \begin{cases} 0, & \text{if the induced subgraph on } N \setminus \{2\} \text{ is connected;} \\ -2, & \text{otherwise,} \end{cases} \\ Exc(N \setminus \{1\}, \text{nuc}(v), v^E) &= \begin{cases} 0, & \text{if the induced subgraph on } N \setminus \{1\} \text{ is connected;} \\ -6, & \text{otherwise.} \end{cases} \end{aligned}$$

Note that, if  $n \geq 4$ ,

$$Exc(N \setminus \{4\}, \text{nuc}(v), v^E) \leq \frac{2}{n-2} - 3 \leq -2 < \frac{-2}{n-2},$$

and, if  $n > 4$ ,

$$Exc(N \setminus \{3\}, \text{nuc}(v), v^E) \leq \frac{2}{n-2} - 2 < -1 < \frac{-2}{n-2}.$$

We may conclude that, if the induced subgraph on  $N \setminus \{1\}$  or the induced subgraph on  $N \setminus \{2\}$  is connected, the highest excess equals 0 and

$$\begin{aligned} B_1(\text{nuc}(v), v^E) &= \{N \setminus \{1\}, N \setminus \{2\}\}, \\ B_1(\text{nuc}(v), v^E) &= \{N \setminus \{1\}\}, \text{ or} \\ B_1(\text{nuc}(v), v^E) &= \{N \setminus \{2\}\}. \end{aligned}$$

Clearly, for these cases,  $B_1(\text{nuc}(v), v^E)$  is not a balanced collection and  $\text{nuc}(v^E) \neq \text{nuc}(v)$ . Note that, if  $n = 4$ , it holds that the induced subgraph on  $N \setminus \{1\}$  or the induced subgraph on  $N \setminus \{2\}$  is connected, due to the connectedness of the graph and the fact that  $\{1, 2\} \notin E$ .

For the remaining case, we can assume that  $n > 4$  and that both induced subgraphs on  $N \setminus \{1\}$  and  $N \setminus \{2\}$  are not connected. Then, the highest excess equals  $\frac{-2}{n-2}$  ( $> -1$ ) and

$$\begin{aligned} B_1(\text{nuc}(v), v^E) &= \{\{j\} \mid j \in N \setminus \{1, 2, 3\}\} \\ &\quad \cup \{\{1, 3\}\} \\ &\quad \cup \{\{1, 2, j\} \mid j \in N \setminus \{1, 2, 3\} \text{ and both } \{1, j\} \in E \text{ and } \{2, j\} \in E\}. \end{aligned}$$

Note that if  $B_1(\text{nuc}(v), v^E) = \{\{j\} \mid j \in N \setminus \{1, 2, 3\}\} \cup \{\{1, 3\}\}$ , then  $B_1(\text{nuc}(v), v^E)$  is not balanced, since  $2 \notin S$  for all  $S \in B_1(\text{nuc}(v), v^E)$ . So let  $j \in N \setminus \{1, 2, 3\}$  be such that  $\{1, 2, j\} \in B_1(\text{nuc}(v), v^E)$ . Suppose  $\lambda : B_1(\text{nuc}(v), v^E) \rightarrow \mathbb{R}_{++}$  is such that  $\sum_{S \in B_1(\text{nuc}(v), v^E): i \in S} \lambda(S) = 1$  for all  $i \in N$ . For  $i = 3$ , this condition boils down to  $\lambda(\{1, 3\}) = 1$ . Then, however,

$$\sum_{S \in B_1(\text{nuc}(v), v^E): 1 \in S} \lambda(S) \geq \lambda(\{1, 3\}) + \lambda(\{1, 2, j\}) > 1.$$

Hence, also in this case,  $\text{nuc}(v^E) \neq \text{nuc}(v)$ . □

Table 2 summarizes the invariance results of this section. For each of the three properties, it identifies the weakest condition on the graph for which invariance of the nucleolus is guaranteed for all communication situations with an underlying game satisfying this property.

Property satisfied by underlying games	Condition on the graph to guarantee invariance of the nucleolus
Strong compromise admissibility	biconnected
Compromise stability	complete
Balancedness	complete

**Table 2** – Survey of invariance of the nucleolus

Interestingly, Theorem 4.2 can be modified if we restrict attention to communication situations with an underlying simple game.

A cooperative game  $v \in TU^N$  is called *simple* if  $v(S) \in \{0, 1\}$  for all  $S \in 2^N$ ,  $v(N) = 1$  and  $v(S) \leq v(T)$  for all  $S, T \in 2^N$  with  $S \subseteq T$ . Moreover, for a simple game  $v \in TU^N$ , the set of *veto-players* is given by

$$\text{veto}(v) = \bigcap \{S \in 2^N \mid v(S) = 1\}.$$

These veto-players play an important role in the computation of the nucleolus. The following proposition relates the set of veto-players to the properties of balancedness, compromise stability and strong compromise admissibility. In addition, it provides a direct, closed formula for the nucleolus for balanced, simple games.

**Proposition 4.1** *Let  $v \in TU^N$  be a simple game. Then the following three statements are equivalent:*

- i)  $\text{veto}(v) \neq \emptyset$ ;
- ii)  $v$  is balanced;
- iii)  $v$  is compromise stable.

Moreover,  $|\text{veto}(v)| = 1$  if and only if  $v$  is strongly compromise admissible. Finally,  $\text{veto}(v) \neq \emptyset$  implies that, for all  $i \in N$ ,

$$\text{nuc}_i(v) = \begin{cases} \frac{1}{|\text{veto}(v)|}, & \text{if } i \in \text{veto}(v); \\ 0, & \text{otherwise.} \end{cases}$$

With regard to the graph-restricted game, note that, if the game underlying a communication situation is simple, then the graph-restricted game is simple too.

We see that the proof of Theorem 4.2 uses a simple game to show that only a complete graph guarantees the invariance of the nucleolus. Interestingly, for the class of communication situations with an underlying balanced and simple game, invariance of the nucleolus can be extended to biconnected graphs.

**Theorem 4.3** *Let  $(N, E)$  be a connected graph. Then,  $\text{nuc}(v^E) = \text{nuc}(v)$  for all  $v \in CS^{N,E}$  with an underlying simple game if and only if  $(N, E)$  is biconnected.*

*Proof:* First consider the ‘if’ part: let  $(N, E)$  be biconnected and let  $v \in CS^{N,E}$  be a communication situation with an underlying simple game. Using Proposition 4.1 it suffices to show that  $\text{veto}(v) = \text{veto}(v^E)$ . Since  $v^E(S) = 1$  implies that  $v(S) = 1$ , it holds that  $\text{veto}(v) \subseteq \text{veto}(v^E)$ . Suppose there exists  $i \in \text{veto}(v^E)$  with  $i \notin \text{veto}(v)$ . Clearly, using monotonicity of a simple game and the fact that  $\text{veto}(v) \neq \emptyset$ , it holds that  $v(N \setminus \{i\}) = 1$ . Then, since the induced subgraph on  $N \setminus \{i\}$  is connected, it holds that  $v^E(N \setminus \{i\}) = v(N \setminus \{i\}) = 1$ . This contradicts the fact that  $i \in \text{veto}(v^E)$ .

Next, the ‘only if’ part: let  $\text{nuc}(v^E) = \text{nuc}(v)$  for all  $v \in CS^{N,E}$  with an underlying simple game. We prove that, if  $(N, E)$  is not biconnected, we can construct a communication situation  $v \in CS^{N,E}$  with an underlying simple game such that  $\text{nuc}(v^E) \neq \text{nuc}(v)$ . Let  $(N, E)$  be not biconnected. Set  $N = \{1, 2, \dots, n\}$  and assume w.l.o.g. that  $\{1, 2\}, \{2, 3\} \in E$ , the induced subgraph on  $N \setminus \{2\}$  is not connected and that players 1 and 3 are in two different components in the induced subgraph on  $N \setminus \{2\}$ . Reconsider  $v \in CS^{N,E}$  with, for all  $S \in 2^N \setminus \{\emptyset\}$ ,

$$v(S) = \begin{cases} 1, & \text{if } \{1, 2\} \subseteq S \text{ or } \{1, 3\} \subseteq S; \\ 0, & \text{otherwise.} \end{cases}$$

Note that  $v$  is a simple game. Recall that, for all  $S \in 2^N \setminus \{\emptyset\}$ ,

$$v^E(S) = \begin{cases} 1, & \text{if } \{1, 2\} \subseteq S; \\ 0, & \text{otherwise,} \end{cases}$$

and hence,  $\text{nuc}(v^E) = (\frac{1}{2}, \frac{1}{2}, 0, \dots, 0) \neq (1, 0, 0, \dots, 0) = \text{nuc}(v)$ . □

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